

Markscheme

November 2019

Mathematics

Higher level

Paper 1

18 pages

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a valid **Method**; working must be seen.
- (M)** Marks awarded for **Method**; may be implied by **correct** subsequent working.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- (A)** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.
- R** Marks awarded for clear **Reasoning**.
- N** Marks awarded for **correct** answers if **no** working shown.
- AG** Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the “must be seen” marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp **A0** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, eg **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (eg substitution into a formula) and **A1** for using the **correct** values.
- Where the markscheme specifies (**M2**), **N3**, etc., do **not** split the marks.

- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final **A1**. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct **FT** working shown, award **FT** marks as appropriate but do not award the final **A1** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	5.65685... (incorrect decimal value)	Award the final A1 (ignore the further working)
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final A1
3.	$\log a - \log b$	$\log(a - b)$	Do not award the final A1

3 N marks

Award **N** marks for **correct** answers where there is **no** working.

- Do **not** award a mixture of **N** and other marks.
- There may be fewer **N** marks available than the total of **M**, **A** and **R** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.

- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (eg $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent A** marks can be awarded, but **M** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

If a candidate incorrectly copies information from the question, this is a misread (**MR**). A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses [**1 mark**].

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (eg $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks (**d**)

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation **DM** should be used and a brief **note** written next to the mark explaining this decision.

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2 \sin(5x - 3)$, the markscheme gives

$$f'(x) = (2 \cos(5x - 3)) 5 (= 10 \cos(5x - 3)) \quad \mathbf{A1}$$

Award **A1** for $(2 \cos(5x - 3)) 5$, even if $10 \cos(5x - 3)$ is not seen.

10 Accuracy of Answers

Candidates should **NO LONGER** be penalized for an accuracy error (**AP**).

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

Section A

1. (a) $p = 1 - \frac{1}{2} - \frac{1}{5} - \frac{1}{5}$ (M1)
 $= \frac{1}{10}$ A1

[2 marks]

(b) attempt to find $E(X)$ (M1)
 $\frac{1}{2} + 1 + 2 + \frac{N}{10} = 10$ A1
 $\Rightarrow N = 65$ A1

Note: Do not allow FT in part (b) if their p is outside the range $0 < p < 1$.

[3 marks]

Total [5 marks]

2. $\frac{1}{2}e^{2x}$ seen (A1)

attempt at using limits in an integrated expression $\left(\left[\frac{1}{2}e^{2x} \right]_0^{\ln k} = \frac{1}{2}e^{2 \ln k} - \frac{1}{2}e^0 \right)$ (M1)

$= \frac{1}{2}e^{\ln k^2} - \frac{1}{2}e^0$ (A1)

Setting their equation = 12 M1

Note: their equation must be an integrated expression with limits substituted.

$\frac{1}{2}k^2 - \frac{1}{2} = 12$ A1

$(k^2 = 25 \Rightarrow) k = 5$ A1

Note: Do not award final A1 for $k = \pm 5$.

[6 marks]

3. attempt to eliminate a variable (or attempt to find $\det A$) **M1**

$$\begin{pmatrix} 2 & -1 & 1 & | & 5 \\ 1 & 3 & -1 & | & 4 \\ 3 & -5 & a & | & b \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 1 & | & 5 \\ 0 & 7 & -3 & | & 3 \\ 0 & -14 & a+3 & | & b-12 \end{pmatrix} \text{ (or } \det A = 14(a-3) \text{)}$$

(or two correct equations in two variables) **A1**

$$\rightarrow \begin{pmatrix} 2 & -1 & 1 & | & 5 \\ 0 & 7 & -3 & | & 3 \\ 0 & 0 & a-3 & | & b-6 \end{pmatrix} \text{ (or solving } \det A = 0 \text{)}$$

(or attempting to reduce to one variable, e.g. $(a-3)z = b-6$) **M1**

$a = 3, b \neq 6$ **A1A1**

[5 marks]

4. attempt to use $\cos(2A + B) = \cos 2A \cos B - \sin 2A \sin B$ (may be seen later) **M1**

attempt to use any double angle formulae (seen anywhere) **M1**

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) **M1**

$$\cos A = \frac{2}{3} \Rightarrow \sin A \left(= \sqrt{1 - \frac{4}{9}} \right) = \frac{\sqrt{5}}{3} \quad \text{(A1)}$$

$$\sin B = \frac{1}{3} \Rightarrow \cos B \left(= \sqrt{1 - \frac{1}{9}} = \frac{\sqrt{8}}{3} \right) = \frac{2\sqrt{2}}{3} \quad \text{A1}$$

$$\cos 2A (= 2 \cos^2 A - 1) = -\frac{1}{9} \quad \text{A1}$$

$$\sin 2A (= 2 \sin A \cos A) = \frac{4\sqrt{5}}{9} \quad \text{A1}$$

$$\text{So } \cos(2A + B) = \left(-\frac{1}{9}\right) \left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right) \left(\frac{1}{3}\right)$$

$$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27} \quad \text{AG}$$

[7 marks]

5. (a) **METHOD 1**

$|z| = \sqrt[4]{4} (= \sqrt{2})$ (A1)

$\arg(z_1) = \frac{\pi}{4}$ (A1)

first solution is $1+i$ A1

valid attempt to find all roots (De Moivre or +/- their components) (M1)

other solutions are $-1+i, -1-i, 1-i$ A1

[5 marks]

METHOD 2

$z^4 = -4$

$(a+ib)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. (M1)

$a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4 = -4$

$(a^4 - 6a^2b^2 + b^4 = -4 \Rightarrow) a = \pm 1$ **and** $(4a^3b - 4ab^3 = 0 \Rightarrow) a = \pm b$ (A1)

first solution is $1+i$ A1

valid attempt to find all roots (De Moivre or +/- their components) (M1)

other solutions are $-1+i, -1-i, 1-i$ A1

[5 marks]

(b) complete method to find area of 'rectangle' (M1)

= 4 A1

[2 marks]

Total [7 marks]

6. $f'(x) = e^{2x} + 2xe^{2x}$ **A1**

Note: This must be obtained from the candidate differentiating $f(x)$.

$= (2^1 x + 1 \times 2^{1-1}) e^{2x}$ **A1**
 (hence true for $n = 1$)

assume true for $n = k$: **M1**
 $f^{(k)}(x) = (2^k x + k2^{k-1}) e^{2x}$

Note: Award **M1** if truth is assumed. Do not allow "let $n = k$ ".

consider $n = k + 1$:

$f^{(k+1)}(x) = \frac{d}{dx} ((2^k x + k2^{k-1}) e^{2x})$

attempt to differentiate $f^{(k)}(x)$ **M1**

$f^{(k+1)}(x) = 2^k e^{2x} + 2(2^k x + k2^{k-1}) e^{2x}$ **A1**

$f^{(k+1)}(x) = (2^k + 2^{k+1} x + k2^k) e^{2x}$

$f^{(k+1)}(x) = (2^{k+1} x + (k+1)2^k) e^{2x}$ **A1**
 $= (2^{k+1} x + (k+1)2^{(k+1)-1}) e^{2x}$

True for $n = 1$ and $n = k$ true implies true for $n = k + 1$.

Therefore the statement is true for all $n (\in \mathbb{Z}^+)$ **R1**

Note: Do not award final **R1** if the two previous **M1s** are not awarded.
 Allow full marks for candidates who use the base case $n = 0$.

[7 marks]

7. (a) attempt to complete the square or multiplication and equating coefficients (M1)
 $2x - x^2 = -(x-1)^2 + 1$ (A1)
 $a = -1, h = 1, k = 1$

[2 marks]

- (b) use of their identity from part (a) $\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{1-(x-1)^2}} dx \right)$ (M1)
 $= \left[\arcsin(x-1) \right]_{\frac{1}{2}}^{\frac{3}{2}}$ or $\left[\arcsin(u) \right]_{-\frac{1}{2}}^{\frac{1}{2}}$ (A1)

Note: Condone lack of, or incorrect limits up to this point.

$$= \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right) \quad (M1)$$

$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) \quad (A1)$$

$$= \frac{\pi}{3} \quad A1$$

[5 marks]

Total [7 marks]

8. a vector normal to Π_p is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ **(A1)**

Note: Allow any scalar multiple of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, including $\begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix}$

attempt to find scalar product (or vector product) of direction vector of line

with any scalar multiple of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ **M1**

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ \sin \theta \\ \cos \theta \end{pmatrix} = 5 \text{ (or } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 5 \\ \sin \theta \\ \cos \theta \end{pmatrix} = \begin{pmatrix} 0 \\ -\cos \theta \\ \sin \theta \end{pmatrix})$$
 A1

(if α is the angle between the line and the normal to the plane)

$$\cos \alpha = \frac{5}{1 \times \sqrt{25 + \sin^2 \theta + \cos^2 \theta}} \text{ (or } \sin \alpha = \frac{1}{1 \times \sqrt{25 + \sin^2 \theta + \cos^2 \theta}})$$
 A1

$$\Rightarrow \cos \alpha = \frac{5}{\sqrt{26}} \text{ or } \sin \alpha = \frac{1}{\sqrt{26}}$$
 A1

this is independent of p and θ , hence the angle between the line and the plane, $(90 - \alpha)$, is also independent of p and θ

R1

Note: The final **R** mark is independent, but is conditional on the candidate obtaining a value independent of p and θ .

[6 marks]

Section B

9. (a) $\cos 105^\circ = \cos(180^\circ - 75^\circ) = -\cos 75^\circ$ **R1**
 $= -q$ **AG**

Note: Accept arguments using the unit circle or graphical/diagrammatical considerations.

[1 mark]

- (b) $AD = CD \Rightarrow \hat{CAD} = 45^\circ$ **A1**
 valid method to find \hat{BAC} **(M1)**
 for example: $BC = r \Rightarrow \hat{BCA} = 60^\circ$
 $\Rightarrow \hat{BAC} = 30^\circ$ **A1**
 hence $\hat{BAD} = 45^\circ + 30^\circ = 75^\circ$ **AG**

[3 marks]

- (c) (i) $AB = r\sqrt{3}$, $AD (= CD) = r\sqrt{2}$ **A1A1**
 applying cosine rule **(M1)**
 $BD^2 = (r\sqrt{3})^2 + (r\sqrt{2})^2 - 2(r\sqrt{3})(r\sqrt{2})\cos 75^\circ$ **A1**
 $= 3r^2 + 2r^2 - 2r^2\sqrt{6}\cos 75^\circ$
 $= 5r^2 - 2r^2q\sqrt{6}$ **AG**

- (ii) $\hat{BCD} = 105^\circ$ **(A1)**
 attempt to use cosine rule on $\triangle BCD$ **(M1)**
 $BD^2 = r^2 + (r\sqrt{2})^2 - 2r(r\sqrt{2})\cos 105^\circ$
 $= 3r^2 + 2r^2q\sqrt{2}$ **A1**

[7 marks]

- (d) $5r^2 - 2r^2q\sqrt{6} = 3r^2 + 2r^2q\sqrt{2}$ **(M1)(A1)**
 $2r^2 = 2r^2q(\sqrt{6} + \sqrt{2})$ **A1**

Note: Award **A1** for any correct intermediate step seen using only two terms.

$$q = \frac{1}{\sqrt{6} + \sqrt{2}}$$

AG

Note: Do not award the final **A1** if follow through is being applied.

[3 marks]

Total [14 marks]

10. (a) (i) attempt to use quotient rule (or equivalent) (M1)

$$f'(x) = \frac{(x^2 - 1)(2) - (2x - 4)(2x)}{(x^2 - 1)^2}$$

$$= \frac{-2x^2 + 8x - 2}{(x^2 - 1)^2}$$
A1

(ii) $f'(x) = 0$ (M1)
 simplifying numerator (may be seen in part (i))
 $\Rightarrow x^2 - 4x + 1 = 0$ or equivalent quadratic equation A1

EITHER
 use of quadratic formula
 $\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$ A1

OR
 use of completing the square
 $(x - 2)^2 = 3$ A1

THEN
 $x = 2 - \sqrt{3}$ (since $2 + \sqrt{3}$ is outside the domain) AG

Note: Do not condone verification that $x = 2 - \sqrt{3} \Rightarrow f'(x) = 0$.
 Do not award the final **A1** as follow through from part (i).

[5 marks]

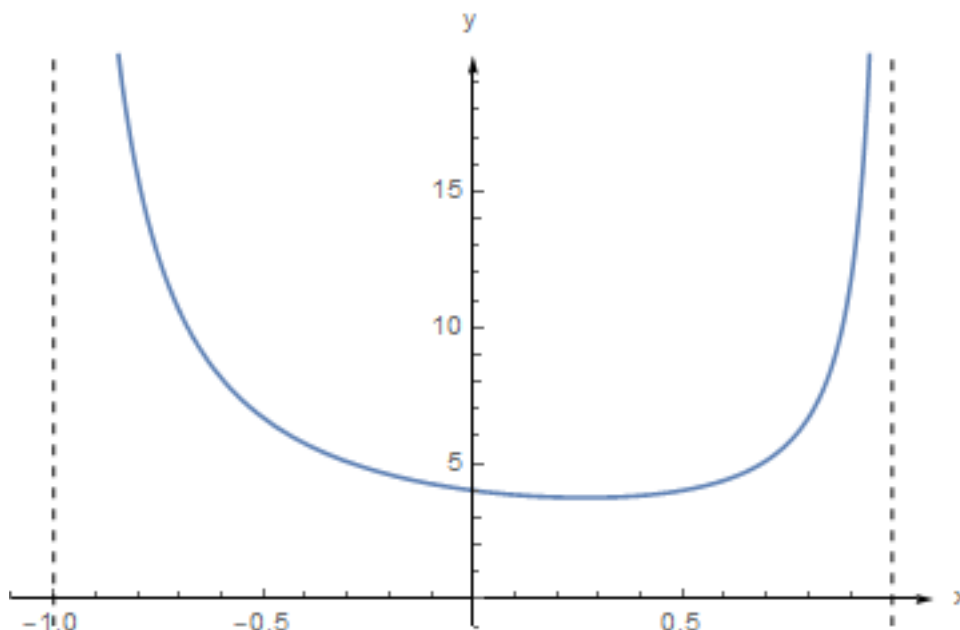
(b) (i) (0, 4) A1

(ii) $2x - 4 = 0 \Rightarrow x = 2$ A1
 outside the domain R1

continued...

Question 10 continued

(iii)



A1A1

award **A1** for concave up curve over correct domain with one minimum point in the first quadrant

award **A1** for approaching $x = \pm 1$ asymptotically

[5 marks]

(c) valid attempt to combine fractions (using common denominator)

$$\frac{3(x-1) - (x+1)}{(x+1)(x-1)}$$

$$= \frac{3x - 3 - x - 1}{x^2 - 1}$$

$$= \frac{2x - 4}{x^2 - 1}$$

M1

A1

AG

[2 marks]

continued...

Question 10 continued

(d)

$$f(x) = 4 \Rightarrow 2x - 4 = 4x^2 - 4$$

M1

$$(x = 0 \text{ or } x = \frac{1}{2})$$

A1

area under the curve is $\int_0^{\frac{1}{2}} f(x) dx$

M1

$$= \int_0^{\frac{1}{2}} \frac{3}{x+1} - \frac{1}{x-1} dx$$

Note: Ignore absence of, or incorrect limits up to this point.

$$= \left[3 \ln|x+1| - \ln|x-1| \right]_0^{\frac{1}{2}}$$

A1

$$= 3 \ln \frac{3}{2} - \ln \frac{1}{2} (-0)$$

$$= \ln \frac{27}{4}$$

A1

area is $2 - \int_0^{\frac{1}{2}} f(x) dx$ or $\int_0^{\frac{1}{2}} 4 dx - \int_0^{\frac{1}{2}} f(x) dx$

M1

$$= 2 - \ln \frac{27}{4}$$

$$= \ln \frac{4e^2}{27}$$

A1

$$\left(\Rightarrow v = \frac{4e^2}{27} \right)$$

[7 marks]

Total [19 marks]

11. (a) (i) $\vec{AV} = \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix}$ **A1**

$$\vec{AB} \times \vec{AV} = \begin{pmatrix} 0 \\ 10 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10(p-10)+10p \\ -10p \\ -10p \end{pmatrix}$$
 A1

$$= \begin{pmatrix} 20p-100 \\ -10p \\ -10p \end{pmatrix} = -10 \begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}$$
 AG

$$\vec{AC} \times \vec{AV} = \begin{pmatrix} 10 \\ 0 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10p \\ 100-20p \\ 10p \end{pmatrix} = 10 \begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix}$$
 A1

(ii) attempt to find a scalar product **M1**

$$-10 \begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet 10 \begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 100(3p^2 - 20p)$$

OR $-\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet \begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 3p^2 - 20p$ **A1**

attempt to find magnitude of either $\vec{AB} \times \vec{AV}$ or $\vec{AC} \times \vec{AV}$ **M1**

$$\left| -10 \begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \right| = \left| 10 \begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} \right| = 10\sqrt{(10-2p)^2 + 2p^2}$$
 A1

$$100(3p^2 - 20p) = 100\left(\sqrt{(10-2p)^2 + 2p^2}\right)^2 \cos \theta$$

$$\cos \theta = \frac{3p^2 - 20p}{(10-2p)^2 + 2p^2}$$
 A1

Note: Award **A1** for any intermediate step leading to the correct answer.

$$= \frac{p(3p - 20)}{6p^2 - 40p + 100}$$
 AG

Note: Do not allow FT marks from part (a)(i).

[8 marks]

continued...

Question 11 continued

- (b) (i) $p(3p - 20) = 0 \Rightarrow p = 0$ or $p = \frac{20}{3}$ **M1A1**
 coordinates are $(0, 0, 0)$ and $\left(\frac{20}{3}, \frac{20}{3}, \frac{20}{3}\right)$ **A1**

Note: Do not allow column vectors for the final **A** mark.

- (ii) two points are mirror images in the plane
 or opposite sides of the plane
 or equidistant from the plane
 or the line connecting the two Vs is perpendicular to the plane **R1**
[4 marks]

- (c) (i) geometrical consideration or attempt to solve $-1 = \frac{p(3p - 20)}{6p^2 - 40p + 100}$ **(M1)**
 $p = \frac{10}{3}, \theta = \pi$ or $\theta = 180^\circ$ **A1A1**

- (ii) $p \rightarrow \infty \Rightarrow \cos \theta \rightarrow \frac{1}{2}$ **M1**
 hence the asymptote has equation $\theta = \frac{\pi}{3}$ **A1**
[5 marks]

Total [17 marks]